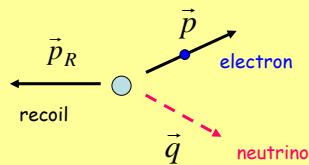


Recall from last class: we calculated the partial decay rate for a final state e^- with momentum \vec{p} using Fermi's Golden Rule:



$$\lambda_{if} = \frac{2\pi}{\hbar} |M_{if}|^2 \rho_f$$

(transitions / sec)

$$\lambda_{if} = G^2 \frac{2\pi}{\hbar c} |M_{nuclear}|^2 \frac{(4\pi)^2}{h^6} p^2 q^2 dp$$

where G is a generic "weak coupling constant" for the process.

- If the electron and neutrino couple to $S = 0$, then it is called a **Fermi decay** and $G = G_V$.
- If they couple to $S = 1$, it is a **Gamow-Teller** decay and $G = G_A$.
- For the **neutron decay**, both $S = 0$ and $S = 1$ configurations are possible, and $S = 1$ is 3 times more likely, so in that case $G^2 = G_V^2 + 3 G_A^2$.

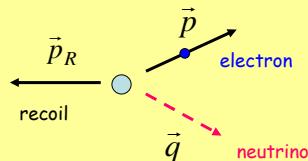
Electron momentum spectrum:

2

$$N(p) dp = N_o \lambda_{if} = (\text{const.}) \times p^2 q^2 dp$$

approx: $K_R = 0$

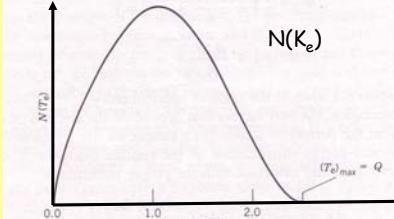
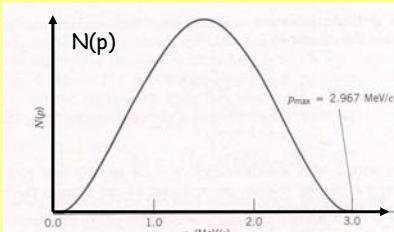
$$\Rightarrow N(p) = (\text{const.}) \times p^2 q^2 = (\text{const}) \times p^2 (Q - K_e)^2$$

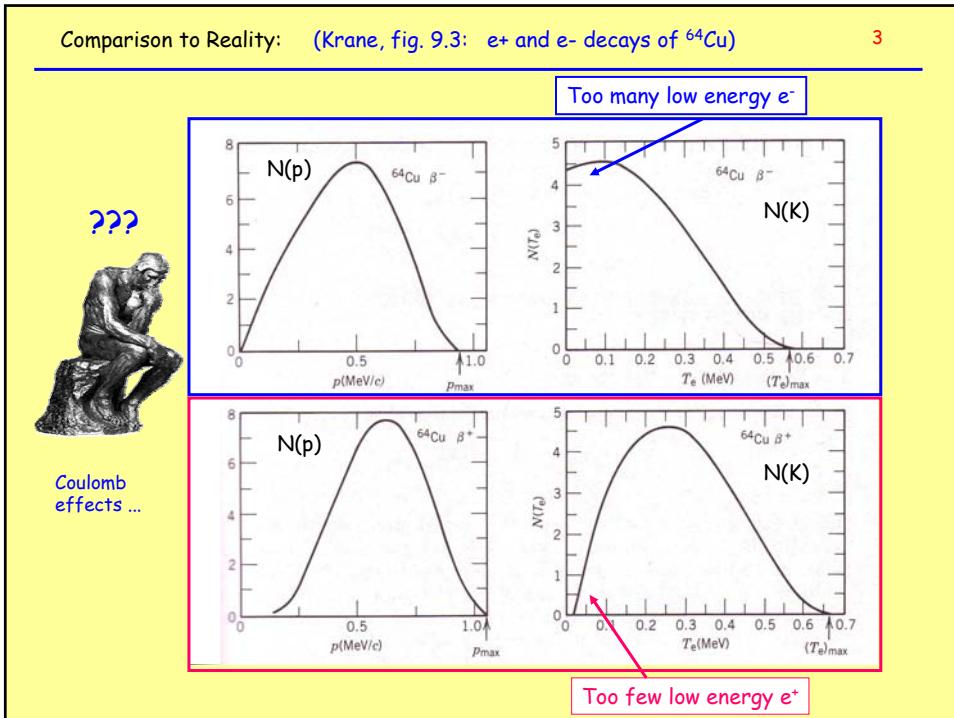


Predicted spectral shapes, Krane, figure 9.2:

(plotted for $Q = 2.5$ MeV, not the neutron!)

(Note: $\max. K(e^-) = Q$)
(Krane uses symbol T for K)





Discrepancy: neglect of Coulomb effects in the final state. 4

Key point: Coulomb distortions of the energy spectra occur AFTER the electron/positron are emitted in the weak decay process.

Modified density of electron/positron states:

$$dn_e = \left(4\pi p^2 dp \frac{V}{h^3} \right) F(Z', p)$$

"Fermi function", depends on the charge Z' of the "daughter nucleus" (final state) and the electron/positron momentum

original result

Approximate correction factor for β^\pm decay:

$$F^\pm(Z', p) \approx \frac{x}{1 - e^{-x}}, \quad x = \mp \frac{2\pi\alpha Z' \sqrt{m^2 + p^2}}{p}, \quad \alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} = \frac{1}{137}$$

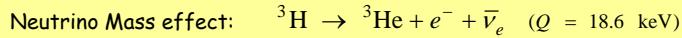
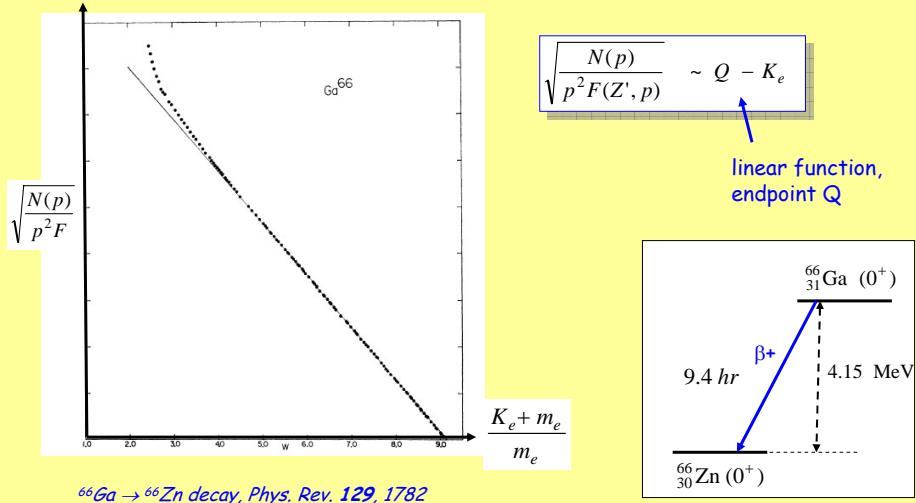
→ Modified electron/positron spectrum prediction:

$$N(p) = C p^2 (Q - K_e)^2 F^\pm(Z', p), \quad C = \frac{G^2}{2\pi^3 \hbar^7 c^3} |M_{nucl}|^2$$

Fermi-Kurie Plot

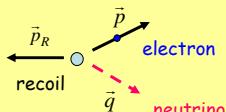
5

Idea: for "allowed decays", corresponding to our approximation: $e^{i\vec{p}_R \cdot \vec{r} / \hbar} = 1$
 inside the nucleus, the electron energy spectrum can be "linearized" if one accounts for the Coulomb distortion via the Fermi function $F(Z', p)$:



6

Idea: shape of the electron energy spectrum near the endpoint (Q) is sensitive to the mass of the electron antineutrino:



recall: $Q \equiv K_R + K_e + K_\nu$

When $K_e \approx Q$, $K_R \approx K_\nu \rightarrow 0$. if $m_\nu \neq 0$, then in this limit, mass effects are most pronounced.

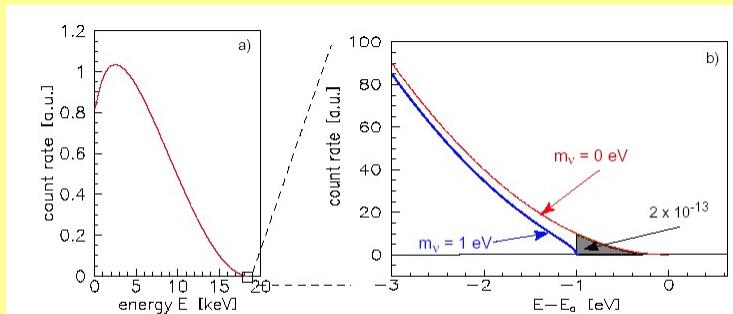


Figure 2: The electron energy spectrum of tritium β decay: (a) complete and (b) narrow region around endpoint E_0 . The β spectrum is shown for neutrino masses of 0 and 1 eV.

figure from "KATRIN" proposal, 2001: (Karlsruhe Tritium Neutrino expt.)

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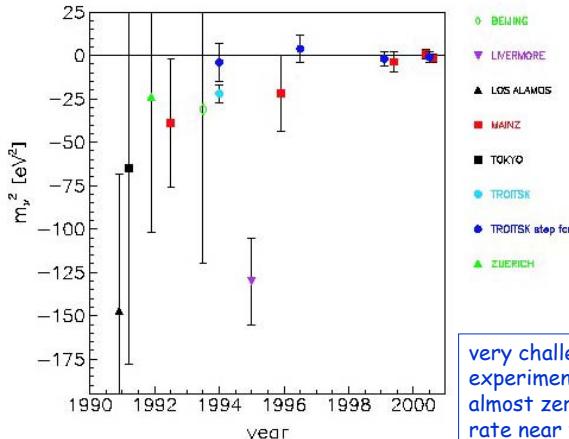


Figure 4: Results of tritium β decay experiments on the observable m_ν^2 over the last decade.

- Best direct upper limit: $m_\nu < 2.2$ eV
- from Sudbury neutrino observatory and other experiments, we have convincing indirect evidence of nonzero neutrino mass that is much smaller than this

Web sites: <http://cupp.oulu.fi/neutrino/nd-mass.html>, <http://www-ik.fzk.de/~katrin/>

Systematic test: total rate for beta decay

8

Our formalism determines λ_{if} , which is the rate (s^{-1}) to a particular final state electron (or positron) momentum p :

$$\lambda_{if}^{\pm}(p) = \frac{G^2}{2\pi^3 \hbar^7 c^3} |M_{nucl}|^2 p^2 (Q - K_e)^2 F^{\pm}(Z', p)$$

(\pm refers to β^{\pm} decay modes)

The total decay rate is obtained by integrating λ_{if} over all allowed e^{\pm} momenta p :

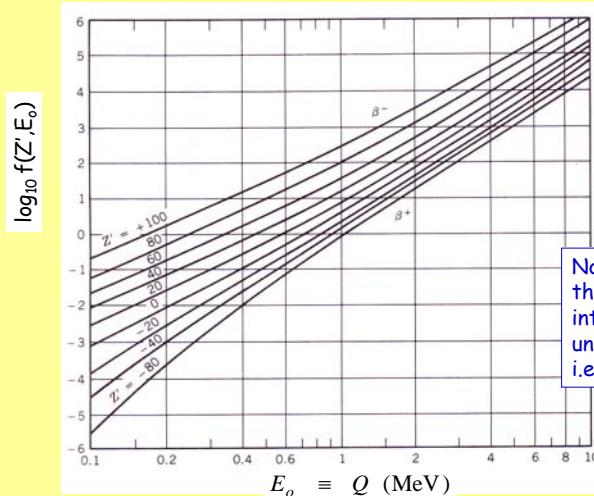
$$\begin{aligned} \lambda(s^{-1}) &= \int_0^{p_{\max}} \lambda_{if}(p) dp = \frac{G^2}{2\pi^3 \hbar^7 c^3} \times |M_{nucl}|^2 \int_0^{p_{\max}} p^2 (Q - K_e)^2 F(Z', p) dp \\ &= (const.) \times |M_{nucl}|^2 f(Z', Q) \end{aligned}$$

"Fermi integral", $f(Z', Q)$

Key point: apart from the nuclear matrix element, the variation in decay rates for different unstable nuclei should only depend on the Fermi integral, which we can calculate independently.

We can use this to test our weak interaction theory!

By convention: $f(Z', E_o) = \frac{1}{m_e^5 c^7} \int_0^{p_{\max}} p^2 (E_o - K_e)^2 F(Z', p) dp, \quad E_o \equiv Q$



By convention, the **half-life**, $t_{1/2} = \tau \ln 2$, with $\tau = 1/\lambda$ is used as a comparison standard for different nuclear beta decays:

we had: $\lambda (s^{-1}) = \frac{G^2}{2\pi^3 \hbar^7 c^3} \times |M_{nucl}|^2 f(Z', Q)$

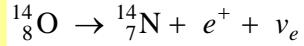
rearranging, we get: $f(Z', Q) t_{1/2} \equiv f t_{1/2} = \ln 2 \times \frac{2\pi^3 \hbar^7}{G^2 m_e^5 c^4 |M_{nucl}|^2}$

dimensionless integral from last slide

Notice: The only difference in the "ft" value between different nuclear beta decays is the value of the nuclear matrix element.

→ If $|M_{nucl}|^2 = 1$ ("superallowed" case in nuclei), the ft values can be used to determine the weak coupling constants $G = (G_V, G_A)$

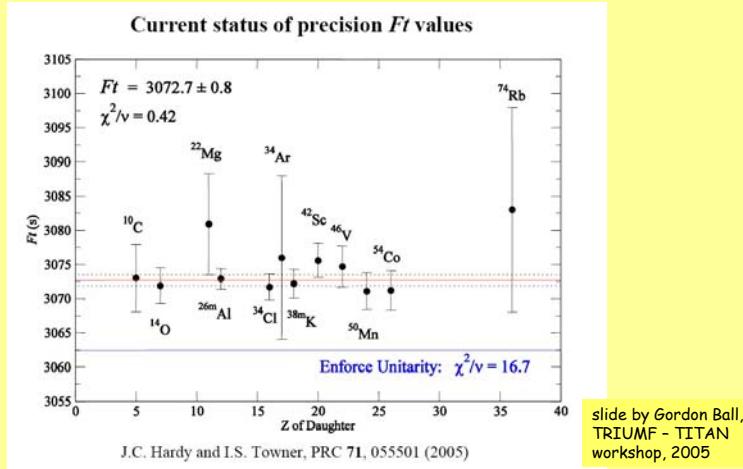
Special case: "superallowed" decays in nuclei with initial and final nuclear states $0^+ \rightarrow 0^+$, e.g.



must have $S = 0$ for the leptons → pure Fermi decay...

Superallowed Fermi $0^+ \rightarrow 0^+$ decays: world's best data for light nuclei

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- all have the same Ft value ~ 3100 sec
- determines the weak coupling constant for Fermi decays:

$$G_V = (1.1358 \pm 0.0004) \times 10^{-5} (\hbar c)^3 / \text{GeV}^2$$

(And $G_A/G_V = -1.25$, more later...)

Can we understand beta decay rates in general?

12

first page of Krane, Appendix C: (symbol ε stands for electron capture/ β^+ decay)

→ 27 isotopes: 8 β^- decays, 6 β^+ decays, spanning 16 orders of magnitude in rate!

Z	A	Abundance			Z	A	Abundance		
		Atomic mass (u)	I^π	or Half-life			Atomic mass (u)	I^π	or Half-life
H	1	1.007825	$\frac{1}{2}^+$	99.985%		10	10.012937	3^+	19.8%
	2	2.014102	1^+	0.015%		11	11.009305	$\frac{3}{2}^-$	80.2%
	3	3.016049	$\frac{1}{2}^+$	12.3 y (β^-)		12	12.014353	1^-	20.4 ms (β^-)
He	3	3.016029	$\frac{1}{2}^+$	$1.38 \times 10^{-4}\%$		13	13.017780	$\frac{3}{2}^-$	17.4 ms (β^-)
	4	4.002603	0^+	99.99986%					
Li	6	6.015121	1^+	7.5%		10	10.016856	0^+	19.2 s (ε)
	7	7.016003	$\frac{1}{2}^-$	92.5%		11	11.011433	$\frac{3}{2}^-$	20.4 m (ε)
	8	8.022486	2^+	0.84 s (β^-)		12	12.000000	0^+	98.89%
Be	7	7.016928	$\frac{3}{2}^-$	53.3 d (ε)		13	13.003355	$\frac{1}{2}^-$	1.11%
	8	8.005305	0^+	0.07 fs (α)		14	14.003242	0^+	5730 y (β^-)
	9	9.012182	$\frac{3}{2}^-$	100 % slowest		15	15.010599	$\frac{1}{2}^+$	2.45 s (β^-)
	10	10.013534	0^+	1.6 My (β^-)					fastest
	11	11.021658	$\frac{1}{2}^+$	13.8 s (β^-)					
B	8	8.024606	2^+	0.77 s (ε)		15	15.000109	$\frac{1}{2}^-$	0.366%
	9	9.013329	$\frac{3}{2}^-$	0.85 as (α)		16	16.006100	2^-	7.13 s (β^-)

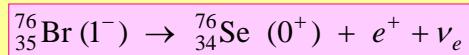
1. According to our theory, the very slow decay: $(1.6 \times 10^6 \text{ yrs})$



should not occur at all, because angular momentum does not add up, i.e.:

$$\vec{0} \neq \vec{3} + (\vec{0} \text{ or } \vec{1})$$

2. Another example: (16.1 hr)



This should not occur because the wavefunctions in the nuclear matrix element have opposite parity, so the integrand is odd and should vanish:

$$M_{nuclear} \equiv \int \psi_f^*(\vec{r}) \psi_i(\vec{r}) d^3r = 0 \quad ???$$

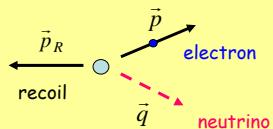
These are two examples of **forbidden decays** - they cannot proceed under the allowed approximation, since

$$M_{if} = G \int \psi_{p,f}^*(\vec{r}) \phi_e^*(\vec{r}) \phi_\nu^*(\vec{r}) \psi_{n,i}(\vec{r}) d^3r = 0 \quad \text{if } \phi_e^*(\vec{r}) \phi_\nu^*(\vec{r}) = \frac{1}{V}$$

Is there some other way they can occur?

Reconsider the electron and antineutrino wave function as a **multipole expansion**:

$$V \phi_e^*(\vec{r}) \phi_\nu^*(\vec{r}) = e^{i\vec{p}_R \cdot \vec{r} / \hbar} \equiv \sum_{L=0}^{\infty} i^L (2L+1) j_L(p_R r / \hbar) P_L(\cos \theta)$$



$$j_L = \text{spherical Bessel Function}$$

$$P_L(\cos \theta) = \text{Legendre polynomial}$$

Beyond the "allowed approximation":

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$$e^{i\vec{p}_R \cdot \vec{r}/\hbar} \equiv \sum_{L=0}^{\infty} i^L (2L+1) j_L(p_R r/\hbar) P_L(\cos\theta)$$

spherical Bessel functions:

$$j_0(x) = \frac{\sin x}{x}; \quad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \dots$$

with $x = p_R r/\hbar$

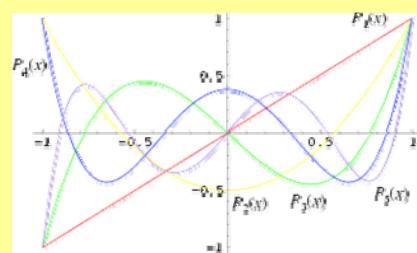
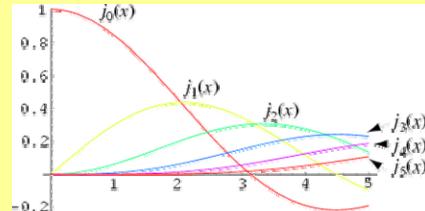
for successively larger L , they become more significant for larger recoil momentum
 \rightarrow this will change the momentum dependence of our prediction!

Legendre polynomials:

$$P_0 = 1, \quad P_1 = x, \quad P_2 = \frac{1}{2}(3x^2 - 1) \dots$$

with $x = \cos\theta$

these introduce a new angular dependence to the integrand for M_{if} \rightarrow equivalent to angular momentum L



Effect on beta transitions:

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- angular momentum coupling for the multipole order L , together with S and nuclear angular momentum allows previously impossible reactions to proceed
- multipole term has parity $(-1)^L$, which allows nuclear states of opposite parity to be "connected" by the beta decay operator
- momentum dependence of the matrix element varies as $(p_R r/\hbar)^L \dots$

since this is small, the lowest multipole order L that satisfies the conservation laws will dominate the transition

rate $\sim |M|^2 \sim (p_R r/\hbar)^{2L} \equiv (0.01)^{2L} \rightarrow$ dramatically smaller for large L

momentum dependence also affects the shape of the spectrum; Kurie plots are not linear unless "shape factors" are taken into account....

- naming convention:

$L = 0$	allowed
$L = 1$	first forbidden
$L = 2$	second forbidden
$L = 3$	third forbidden....

Classification: all known decays....

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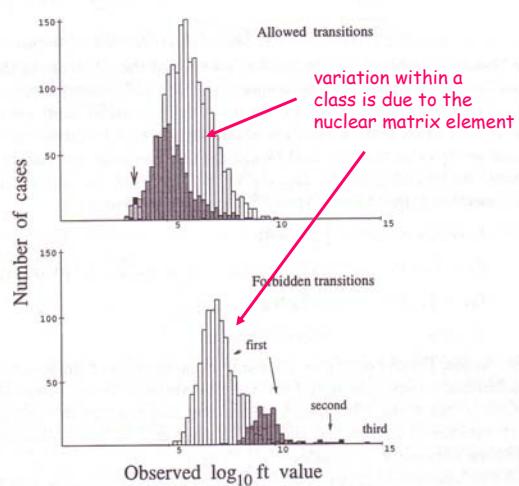
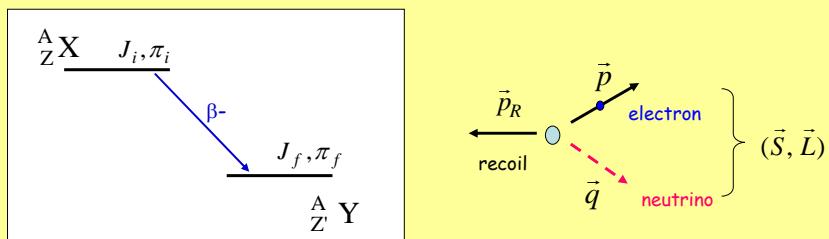


Figure 5-8: Systematics of observed $\log ft$ values. The grey area in the upper panel shows 718 cases of $0^+ \Rightarrow 1^+$ allowed transitions, and the remaining 1741 cases of other allowed decays are shown by the white histogram. The peak of the distribution for the 24 cases of $0^+ \rightarrow 0^+$ superallowed decay is indicated by the arrow.

Classification of decays: general

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Nuclear case: $\frac{A}{Z} X \rightarrow \frac{A}{Z'} Y + e^- + \bar{\nu}_e$



$$\vec{J}_i = \vec{J}_f + \vec{S} + \vec{L}$$

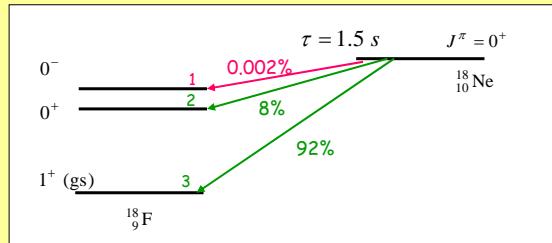
$$\pi_i = \pi_f (-1)^L$$

with $S = 0$ (Fermi) or $S = 1$ (Gamow-Teller)

Smallest value of L that is consistent with conservation laws will dominate the transition.

Example: β^+ decay of ^{18}Ne

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Branching ratio (BR): the fraction of decays that go to a particular final state.

In this case, $\lambda_{\text{total}} = 1/\tau = 0.667 \text{ sec}^{-1}$; $\lambda = \lambda_1 + \lambda_2 + \lambda_3$, with $\lambda_i = \text{BR}(i) \lambda_{\text{total}}$

Transition 1: $0^+ \rightarrow 0^-$ This is a **first forbidden GT decay**, with the slowest partial rate:

$$\vec{0} = \vec{0} + \vec{S} + \vec{L}; \quad (+) = (-) \times (-1)^L \rightarrow L = 1, S = 1$$

Transition 2: $0^+ \rightarrow 0^+$ This is an **allowed Fermi decay**:

$$\vec{0} = \vec{0} + \vec{S} + \vec{L}; \quad (+) = (+) \times (-1)^L \rightarrow L = 0, S = 0$$

Transition 3: $0^+ \rightarrow 1^+$ This is an **allowed Gamow-Teller decay**

$$\vec{0} = \vec{1} + \vec{S} + \vec{L}; \quad (+) = (+) \times (-1)^L \rightarrow L = 0, S = 1$$